Homework 9
Real Analysis

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Proposition 0.1 (Exercise 9)**.** *Let F be a closed subset of $\mathbb{R}$ and define $\delta : \mathbb{R} \to \mathbb{R}$ by*

$$\delta(x) = d(x, F) - \inf\{|x - y| : y \in F\}$$

Then

$$\lim_{|y| \to 0} \frac{\delta(x + y)}{|y|} = 0$$

for almost every $x \in F$.

Proof. Suppose that x is a Lebesgue density point of F. Then x is in the closure of F, so $\delta(x) = 0$. Obviously, F covers at least $m(F \cap [x, x + |y|])$ of $[x, x + |y|]$, so there is some point in F within at least distance $|y| - m(F \cap [x, x + y])$ of $x + |y|$, so

$$\delta(x + |y|) \leq |y| - m(F \cap [x, x + |y|])$$

If we use $[x, x + y]$ to mean $[x + y, x]$ when $y < 0$, we have more the more general estimate

$$\delta(x + y) \leq |y| - m(F \cap [x, x + y])$$

Now we have

$$\lim_{|y| \to 0} \frac{\delta(x + y)}{|y|} \leq \lim_{|y| \to 0} \frac{|y| - m(F \cap [x, x + y])}{|y|} = \lim_{|y| \to 0} \left(\frac{|y|}{|y|} - \frac{m(F \cap [x, x + y]}{m([x, x + y]} \right)$$

Since x is a point of Lebesgue density of F,

$$\lim_{m([x, x+y]) \to 0} \frac{m(F \cap [x, x + y])}{m([x, x + y])} = 1$$

so

$$\lim_{|y| \to 0} \frac{\delta(x + y)}{|y|} \leq \lim_{|y| \to 0} (1 - 1) = \lim_{|y| \to 0} 0 = 0$$

Since $\delta(x + y) \geq 0$ for all x, y, each term $\delta(x + y)/|y|$ is non-negative. Thus if the limit is less than or equal to zero, it must be zero. $\square$

Proposition 0.2 (Exercise 9)**.** *There exists a function $f : \mathbb{R} \to \mathbb{R}$ such that f is increasing, f is discontinuous on $\mathbb{Q}$, and f is continuous on $\mathbb{R} \setminus \mathbb{Q}$.*

Proof. Define a function

$$h(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

Fix an enumeration $\{q_n\}_{n=1^\infty}$ of $\mathbb{Q}$. We define a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \sum_{k=1}^{n} 2^{-k} h(x - q_k)$$

Intuitively speaking, we introduce a jump of height 2^{-k} at the kth rational. Since $0 \leq h(x - q_k) \leq 1$ for all x, k, we have the bounds $0 \leq f_n(x) \leq 1$ for all x. Notice that f_n is increasing (with respect to n), so it converges to a pointwise limit at each x. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \lim_{n\to\infty} f_n(x)$$

We claim that f is increasing, discontinuous on $\mathbb{Q}$, and continuous on $\mathbb{R} \setminus \mathbb{Q}$. First, it is increasing because each $h(x - q_k)$ is increasing so each f_n is increasing. Then the pointwise limits must also be increasing, so f is increasing. Second, f is discontinuous at each $q \in \mathbb{Q}$ by construction. There is a jump discontinuity of heigh 2^{-k} at q_k. We have

$$\lim_{x\to q_k^+} f(x) \geq f(q_k)$$

But

$$\lim_{x\to q_k^-} f(x) \leq f(q_k) - 2^{-k}$$

so f cannot be continuous at q_k. Finally, we claim that f is continuous at $x \in \mathbb{R} \setminus \mathbb{Q}$. Fix $x_0 \in \mathbb{R} \setminus \mathbb{Q}$ and let $\epsilon > 0$. Choose $n \in \mathbb{N}$ large enough that $2^{-n} < \epsilon$, then choose $\delta = \frac{1}{2}\min\{q_1, \ldots, q_n\}$. Because x_0 is not equal to any $q_1, \ldots, q_n$, we then have $(x_0 - \delta, x_0 + \delta) \cap \{q_1, \ldots, q_n\} = \emptyset$, so all of the jumps of heights $2^{-1}, \ldots, 2^{-n}$ happen outside the interval $(x_0 - \delta, x_0 + \delta)$. So the maximum increase of f on this interval is the sum of all other jumps, that is,

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \sum_{k=n+1}^{\infty} 2^{-k} = 2^{-n} < \epsilon$$

and since we chose n large enough that 2^{-n}, we have the required estimate for $|f(x) - f(x_0)|$. Thus f is continuous at x_0.

□

Proposition 0.3 (Exercise 11, part one)**.** *For $a, b > 0$ define*

$$f(x) = \begin{cases} x^a \sin(x^{-b}) & 0 < x \leq 1 \\ 0 & x = 0 \end{cases}$$

Then f is of bounded variation on $[0, 1]$ if and only if $a > b$.

Proof. First suppose that $a > b$. For $x \neq 0$,

$$f'(x) = ax^{a-1}\sin(x^{-b}) - bx^{a-b-1}\cos(x^{-b})$$

Recall that the Reimann integral $\int_0^1 |f'(x)|$ is defined as a limit of Riemann sums over all partitions. Note that for any partition $0 = t_0 < \ldots < t_N = 1$,

$$\sum_{k=1}^{N} |f(x_k) - f(x_{k-1})| \leq \int_0^1 |f'(x)|dx$$

where the integral on the right is a Riemann integral. Thus the total variation of F, which is the supremum over such partitions, is bounded above as well.

$$T_f(a,b) = \sup \sum_{k=1}^{N} |f(x_k) - f(x_{k-1})| \leq \int_0^1 |f'(x)|dx$$

On $(0,1]$, the function f is continuous, so by the fundamental theorem of calculus,

$$\begin{aligned}
\int_0^1 |f'(x)| &\leq \int_0^1 |ax^{a-1}\sin(x^{-b}) - bx^{a-b-1}\cos(x^{-b})|dx \\
&\leq \int_0^1 |ax^{a-1}\sin(x^{-b})|dx + \int_0^1 |bx^{a-b-1}\cos(x^{-b})|dx \\
&\leq \int_0^1 ax^{a-1}dx + \int_0^1 bx^{a-b-1}dx \\
&= 1 + \frac{b}{a-b} < \infty
\end{aligned}$$

Note that we used the fact that $a, b > 0$ and $a - b > 0$ in integrating. Thus f is of bounded variation on $[0,1]$. Now suppose that $a \leq b$. We will show that f is not of bounded variation. Define

$$t_{2k} = (2\pi k + \pi/2)^{-1/b} \qquad t_{2k+1} = (2\pi k)^{-1/b}$$

As a preliminary computation,

$$\begin{aligned}
f(t_{2k}) &= (2\pi k + \pi/2)^{-a/b}\sin(2\pi k + \pi/2) = (2\pi k + \pi/2)^{-a/b} \\
f(t_{2k+1}) &= (2\pi k)^{-a/b}\sin(2\pi k) = 0 \\
|f(t_{2k+1}) - f(t_{2k})| &= |f(t_{2k})| = (2\pi k + \pi/2)^{-a/b} \\
|f(t_{2k}) - f(t_{2k-1})| &= |f(t_{2k})| = (2\pi k + \pi/2)^{-a/b}
\end{aligned}$$

Then the variation of f on this partition is

$$\sum_{n=1}^{N} |f(t_n) - f(t_{n-1})| = \sum_{k=1}^{N} (2\pi k + \pi/2)^{-a/b}$$

The total variation is the at least as big as the limit as $N \to \infty$ of this sum,

$$T_f(a,b) \geq \sum_{k=1}^{\infty} (2\pi k + \pi/2)^{-a/b}$$

We assumed that $a \leq b$, so $a/b \leq 1$. Thus the series diverges by comparison with a p-series, so f does not have bounded variation on $[0,1]$. □

Proposition 0.4 (Exercise 12, part one)**.** *Define*

$$F(x) = \begin{cases} x^2 \sin \frac{1}{x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

on $[0,1]$*. Then* $F'(x)$ *exists everywhere.*

Proof. For $x \neq 0$, we can use standard differentiation techniques to get

$$F'(x) = 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2}$$

which exists for all $x \neq 0$. At $x = 0$, we apply the definition of the derivative to get

$$F'(0) = \lim_{h \to 0} \frac{F(h) - F(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin \frac{1}{h^2}}{h} = \lim_{h \to 0} h \sin \frac{1}{h^2}$$

We have $-h \leq h \sin \frac{1}{h^2} \leq h$ for $h \neq 0$, so by the squeeze theorem,

$$\lim_{h \to 0} h \sin \frac{1}{h^2} = 0$$

so $F'(0) = 0$. Hence $F'(x)$ exists for all $x \in [0,1]$. □

Proposition 0.5 (Exercise 12, part two)**.** *Let* F *be as in the above proposition.* $F'(x)$ *is not integrable on* $[-1,1]$*.*

Proof. First, note that it is sufficient to show that $F'(x)$ is not integrable on $[0,1]$, because we have the symmetry $|F'(x)| = |F'(-x)|$, thus

$$\int_{-1}^{1} |F'(x)| dx = 2 \int_{-1}^{1} |F'(x)| dx$$

So it is sufficient to show that the integral from 0 to 1 is unbounded. In general for real numbers a, b we have $|b| - |a| \leq |a - b|$, so for $x \neq 0$ we have

$$\left| \frac{2}{x} \cos \frac{1}{x^2} \right| - \left| 2x \sin \frac{1}{x^2} \right| \leq |F'(x)|$$

and on the interval $[0,1]$ we have $-2 \leq -|2x| \leq -|2x \sin \frac{1}{x^2}|$, so

$$\left| \frac{2}{x} \cos \frac{1}{x^2} \right| - 2 \leq \left| \frac{2}{x} \cos \frac{1}{x^2} \right| - \left| 2x \sin \frac{1}{x^2} \right| \leq |F'(x)|$$

So to show that $\int_0^1 |F'(x)| dx$ is infinite, we just need to show that

$$\int_0^1 \left| \frac{2}{x} \cos \frac{1}{x^2} \right| dx = 2 \int_0^1 \frac{1}{x} \left| \cos \frac{1}{x^2} \right| dx = \infty$$

Using Mathematica, we find that $|\cos \frac{1}{x^2}| > \frac{1}{2}$ on the disjoint intervals

$$E_n = (a_n, b_n) = \left(\left(\frac{3}{\pi(6n+1)} \right)^{1/2}, \left(\frac{3}{\pi(6n-1)} \right)^{1/2} \right)$$

Then we define $E = \bigcup_{n=1}^{\infty} E_n$. Since $|\cos(1/x^2)| > \frac{1}{2}$ on E, we have $\frac{1}{2}\chi_E \leq |\cos \frac{1}{x^2}|$. Then

$$\int_0^1 \frac{1}{x} \left| \cos \frac{1}{x^2} \right| dx \geq \int_0^1 \frac{1}{2} \chi_E(x) \frac{1}{x} dx = \frac{1}{2} \int_E \frac{1}{x} dx = \frac{1}{2} \sum_{n=1}^{\infty} \int_{E_n} \frac{1}{x} dx = \frac{1}{x} \sum_{n=1}^{\infty} \int_{a_n}^{b_n} \frac{1}{x} dx$$
$$= \frac{1}{2} \sum_{n=1}^{\infty} \log(b_n) - \log(a_n) = \frac{1}{2} \sum_{n=1}^{\infty} \log\left(\frac{b_n}{a_n} \right) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{2} \log\left(\frac{6n+1}{6n-1} \right)$$

This final sum is divergent since the terms do not even converge to zero. Thus our integral diverges, so $F'(x)$ is not integrable on $[-1, 1]$. □

Proposition 0.6 (Exercise 14a). *Let F be a continuous function on $[a, b]$. Then define*

$$D^+(F)(x) = \limsup_{h \to 0,\, h > 0} \frac{F(x+h) - F(x)}{h}$$

The function $D^+(F)$ is measurable.

Proof. We need to show that for $\alpha \in \mathbb{R}$, the set

$$E_\alpha = \{x \in \mathbb{R} : D^+(F)(x) > \alpha\}$$

is measurable. Define $G_\alpha(x) = F(x) - \alpha x$. Then

$$\begin{aligned} D^+(F)(x) > \alpha &\iff F(x+h) - F(x) > h\alpha && \text{for some } h > 0 \\ &\iff F(x+h) - x\alpha - h\alpha > F(x) - x\alpha && \text{for some } h > 0 \\ &\iff G_\alpha(x+h) > G_\alpha(x) && \text{for some } h > 0 \end{aligned}$$

Thus

$$E_\alpha = \{x \in \mathbb{R} : \text{ there exists } h > 0 \text{ so that } G_\alpha(x+h) > G_\alpha(x)\}$$

Note that since F is continuous, so is G_α. Thus by the rising sun lemma (Lemma 3.5 in Stein and Shakarchi), E_α is open or empty for each α. Thus it is always measurable, so $D^+(F)$ is a measurable function. □

Proposition 0.7 (Exercise 14b). *Let F be a bounded increasing function on $[a, b]$ and let $J(x) = \sum_{n=1}^{\infty} a_n j_n(x)$ be the jump function associated to F. Then*

$$\limsup_{h \to 0} \frac{J(x+h) - J(x)}{h}$$

is measurable.

Proof. For $N \in \mathbb{N}$ define

$$J_N(x) = \sum_{n=1}^{N} \alpha_n j_n(x)$$

and then for $k, m, N \in \mathbb{N}$ with $k > m$, define

$$F_{k,m}^N(x) = \sup_{\frac{1}{k} \leq |h| \leq \frac{1}{m}} \left| \frac{J_N(x+h) - J_N(x)}{h} \right|$$

Then since $F_{k,m}^N$ is measurable,

$$\lim_{N \to \infty} F_{k,m}^N = \sup_{\frac{1}{k} \leq |h| \leq \frac{1}{m}} \left| \frac{J(x+h) - J(x)}{h} \right|$$

is measurable. Now we take the limit as $k \to \infty$, to get that

$$\lim_{k \to \infty} \sup_{\frac{1}{k} \leq |h| \leq \frac{1}{m}} \left| \frac{J(x+h) - J(x)}{h} \right| = \sup_{0 \leq |h| \leq \frac{1}{m}} \left| \frac{J(x+h) - J(x)}{h} \right|$$

is measurable. Finally, taking the limit as $m \to \infty$, we get

$$\lim_{m \to \infty} \sup_{0 \leq |h| \leq \frac{1}{m}} \left| \frac{J(x+h) - J(x)}{h} \right| = \limsup_{h \to \infty} \left| \frac{J(x+h) - J(x)}{h} \right|$$

is measurable, as claimed. □